My Overview of Structural dependencies in Discrete and Continuous time Markov processes

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The study of stochastic processes in this module begins with discrete time Markov chains and gradually builds a model that applies to continuous time processes. Understanding this progression makes it possible to explain how later results rely on earlier material. In my essay I explore the natural way to trace this progression, which is to follow the ideas of communicating classes, irreducibility and periodicity from discrete time into the analysis of Poisson processes and the construction of continuous time Markov jump processes.

A discrete time Markov chain is determined by its transition matrix $P = (p_{ij})$ on a countable state space S. Once this is established, the behaviour of the chain depends on how states reach one another. The formal expression of this is accessibility. State j is accessible from i if there exists $n \geq 0$ such that the n step transition probability satisfies

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j \mid X_0 = i) > 0.$$

If this can occur in both directions then i communicates with j. The equivalence relation formed by communication partitions S into communicating classes. This classification gives the first important interpretation of the long term structure of a chain because many later results depend on identifying whether the chain is contained within one class or split across several. For example, if the chain decomposes into multiple classes then any argument about convergence to a single stationary pattern must take this split into account.

Irreducibility appears naturally here. A chain is irreducible if its state space is a single communicating class, so every state is accessible from every other state. This condition is critical for the sections that follow because it determines whether properties apply to the entire chain rather than to a restricted subset. For instance, theorems involving periods, hitting times and limiting distributions often assume irreducibility. Without it, any statement about the chain's behaviour must be applied class by class. Examples such as the simple random walk on $\mathbb{Z}$ with $0 < p < 1$ and the gambler's ruin model illustrate how irreducibility affects analysis. The simple random walk is irreducible, whereas the gambler's ruin chain is not, since the boundary states form closed classes.

Periodicity, another structural property of discrete chains, depends on the set of return times to a state. For a state i, the period is defined as

$$d_i = \gcd\{n \geq 1 : p_{ii}^{(n)} > 0\}.$$

Within a communicating class all states share the same period. This becomes essential when discussing convergence. If an irreducible chain is periodic with $d > 1$ then the n step transition probabilities oscillate between subsets indexed by $n \bmod d$, and the chain cannot approach a single limiting distribution along all integers n. Later results work only when the chain is aperiodic, with $d_i = 1$, as this is a standard assumption in many limit theorems.

These first ideas influence a lot of the later discrete time material. Statements about hitting probabilities or expected hitting times implicitly use whether certain states are even reachable.

If a class C is closed then, once the chain enters C, no future transition can leave it, and so every hitting time relating to states outside C is infinite. An absorbing state is a single closed state with $p_{ii} = 1$ and no transitions to others. This property continues to matter in later proofs, even if not stated directly.

When moving to continuous time, we construct processes that keep the same kind of structure. The Poisson process is the simplest continuous time model with independent and stationary increments. It counts arrivals over time, beginning from $X(0) = 0$. For $s, t \geq 0$ the increment satisfies

$$X(t+s) - X(t) \sim \mathrm{Po}(\lambda s),$$

and increments over disjoint time intervals are independent. The independence of increments ensures that the behaviour of the process after time t does not depend on earlier increments, which matches the Markov property in continuous time.

The Poisson process connects back to the discrete ideas through its jump structure. This is seen by rewriting the process in terms of exponential holding times. Let T_1 be the time of the first arrival. Then

$$\mathbb{P}(T_1 > t) = \mathbb{P}(X(t) = 0) = e^{-\lambda t},$$

so $T_1 \sim \mathrm{Exp}(\lambda)$. Repeating this gives a sequence of exponential holding times that are all independent and identically distributed. The jump chain of the Poisson process then moves deterministically from n to $n+1$ at jump times $J_n = T_1 + \cdots + T_n$. The exponential distribution has the memoryless property

$$\mathbb{P}(T > t + s \mid T > t) = \mathbb{P}(T > s),$$

which makes the Poisson process fit the Markov idea in continuous time, just as accessibility does in discrete time.

Continuous time Markov jump processes are built by pairing a discrete jump chain with exponential holding times set by the generator matrix $Q = (q_{ij})$. For each state i, the off diagonal entries $q_{ij} \geq 0$ for $j \neq i$ represent transition rates to j, and the total rate out of i is

$$q_i = \sum_{j \neq i} q_{ij},$$

with diagonal entries $q_{ii} = -q_i$. The holding time in state i is exponential with parameter q_i. When a jump occurs from i, the probability that it goes to $j \neq i$ is

$$r_{ij} = \frac{q_{ij}}{q_i},$$

so the embedded jump chain (Y_n) has transition matrix $R = (r_{ij})$. Earlier concepts from discrete time reappear here. If the rates allow movement in both directions between i and j along some path, then i and j communicate in the jump chain. If there is a path that leaves a subset and never returns, that subset is not closed. If $q_i = 0$ for all outgoing transitions except i itself, then i is absorbing. This idea is reused when analysing hitting times or convergence.

This makes it increasingly clear that continuous time theory is built on discrete time ideas. Communicating classes, irreducibility and periodicity classify the behaviour of the jump chain, and since the continuous time process moves according to this chain the same classification applies. The holding times change how long the process stays in each state but not the transitions themselves. This lets us reuse many discrete time arguments by translating them into continuous time through the generator Q and the exponential holding times.

Overall, there is a clear progression from discrete chains to the Poisson process and then to continuous time jump processes. The early ideas of accessibility, communicating classes, irreducibility and periodicity give the basic structure needed to understand any Markov process on a countable state space. The Poisson process is the first continuous time example where these ideas still apply, and the general jump process shows how they extend once we introduce a generator matrix. Later topics such as hitting times, expected times to absorption and long term behaviour use the same principles. This makes the discrete and continuous parts of the module evident as connected steps of the same subject rather than separate areas.

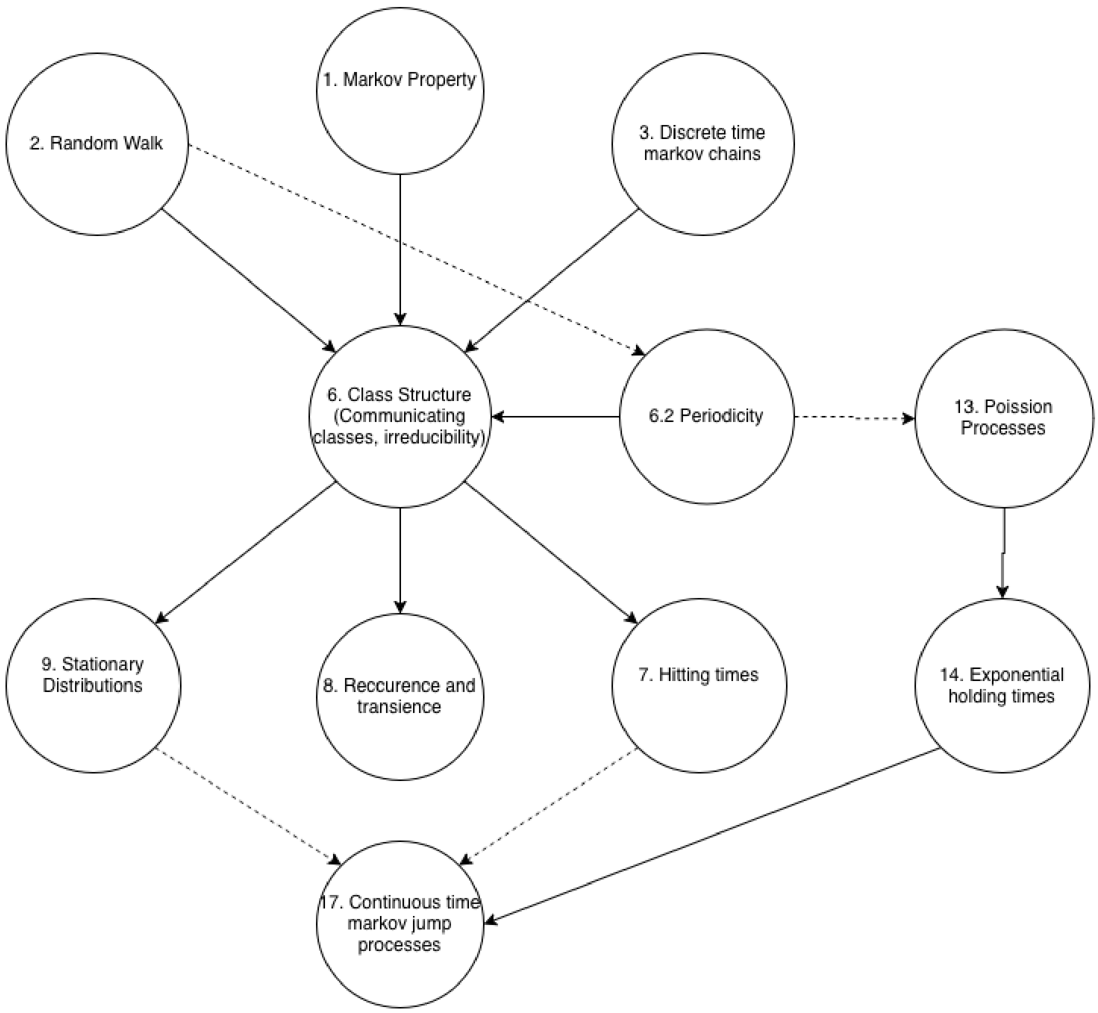

Figure 1: My radial dependency diagram.